AIAA JOURNAL
Vol. 37, No. 7, July 1999

Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Improved First-Order Approximation of Eigenvalues and Eigenvectors"

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THE authors of Ref. 1 present a method for using a reduced basis to better approximate eigenvalues and eigenvectors that Murthy and Haftka have previously published as the reduction method approximation.[2] Equations (9–11) in Ref. 1 are equivalent to Eqs. (8–10) in Ref. 2. They differ in notation and in the use of real, symmetric matrices for self-adjoint structural problems in Ref. 1, compared with the use of complex, general matrices for non-self-adjoint problems in Ref. 2. Hence, the method of Ref. 1 is a special case of the reduction method approximation. In contrast to the title of Ref. 1, the authors of Ref. 2 demonstrate that the reduction method is actually a third-order method.

The two references differ in one further aspect, namely, in selecting the root of the resulting 2×2 reduced-basis, linear eigenvalue problem. In Ref. 1 the authors use five heuristic criteria, whereas Murthy and Haftka choose the root closest to the linear (first-order) Taylor-series approximation of the eigenvalue (comparable to the second of the five criteria in Ref. 1).

An observation may improve slightly the heuristic algorithm for determining the eigenvector approximation. The reduction-method approximation contains all of the ingredients to calculate the second-order Taylor-series approximation (TSA2) of the eigenvalue. The TSA2 of the eigenvalue

$$\lambda_i \cong \lambda_i^0 + \sum_{j=1}^{p}\left(\frac{\partial \lambda_i}{\partial x_j}\right)\Delta x_j + \frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{p}\left(\frac{\partial^2 \lambda_i}{\partial x_j \partial x_k}\right)\Delta x_j \Delta x_k \tag{1}$$

requires second derivatives of the eigenvalue with respect to design variables. However, the second derivative, in turn, requires only first-order derivatives of the eigenvalue, mass, stiffness, and associated eigenvector, all of which are available in Eqs. (6–8) of Ref. 1. Following the notation of Ref. 1, the eigenvalue second derivative used in Eq. (1) may be expressed as[3]

$$\frac{\partial^2 \lambda_i}{\partial x_j \partial x_k} = \left[\{\phi^0\}_i^T([F_{i,j}]\{\phi_{i,k}\} + [F_{i,k}]\{\phi_{i,j}\}) - \left(\frac{\partial \lambda_i}{\partial x_j}\{\phi^0\}_i^T\frac{\partial [M^0]}{\partial x_k} + \frac{\partial \lambda_i}{\partial x_k}\{\phi^0\}_i^T\frac{\partial [M^0]}{\partial x_j}\right)\{\phi^0\}_i\right] \Big/ \{\phi^0\}_i^T[M^0]\{\phi^0\}_i \tag{2}$$

where

$$[F_{i,j}] = \frac{\partial [K^0]}{\partial x_j} - \lambda_i \frac{\partial [M^0]}{\partial x_j} - \frac{\partial \lambda_i}{\partial x_j}[M^0] \tag{3}$$

Knowing the TSA2 of the eigenvalue can simplify the selection of the appropriate root for the reduced-basis scaling parameters in Eq. (9) of Ref. 1. The root closest to the TSA2 of the eigenvalue should be chosen for use in Eq. (19) of Ref. 1. This suggests an interpretation for the reduced-basis eigenproblem. Substituting the current Eq. (1) into Eq. (11) of Ref. 1 does not, in general, allow a nontrivial solution of the 2×2 reduced-basis equations for the scaling parameters of the perturbed eigenvector. Choosing the root of Eq. (4) of Ref. 1 closest to λ_i in Eq. (1) implies an associated eigenvector approximation in the space spanned by the unperturbed eigenvector and its first-order differential, i.e., Eq. (5) of Ref. 1. Thus, the reduced basis, consisting of the zero- and first-order eigenvectors, approximates the TSA2 of the eigenvector. The corresponding root represents a third-order approximation of the eigenvalue.

References

[1]Nair, P. B., Keane, A. J., and Langley, R. S., "Improved First-Order Approximation of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 36, No. 9, 1998, pp. 1721–1727.

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Received 19 October 1998; accepted for publication 28 January 1999.

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Reply by the Authors to R. A. Canfield

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THE authors appreciate the comments made by Canfield and his bringing to our notice the equivalence between the method presented in Ref. 1 with an earlier study published in Ref. 2.

Before addressing the issues raised by Canfield, the authors would like to reemphasize some of the motivations behind Ref. 1. The main objective of the work was to present a general method for approximating the eigenvalues and eigenvectors of modified structural dynamical systems. In contrast to most earlier studies, an important focus was to develop a method that could be applied to approximate the low-medium-frequency eigenmodes for moderate to large magnitudes of perturbations in the system matrices, which we needed

Received 2 January 1999; accepted for publication 28 January 1999.

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when using modern stochastic search methods. The term *improved first-order approximation* was used in the title to indicate clearly that only the first-order derivative information is used in the method presented. The central idea of our formulation was to make use of the baseline eigenvector and its first-order perturbation for each eigenmode to construct a sequence of reduced eigenproblems. The idea of using first-order derivatives as high-quality basis vectors is not new; see, for example, Ref. 3 for some successful applications of this concept to structural reanalysis problems. As pointed out by Noor and Whitworth,[3] such reduced-basis approximation methods are equivalent to the classical Bubnov–Galerkin technique. However, because different basis vectors are applied to reanalyze each eigenmode in our method, the classical Bubnov–Galerkin perspective is useful only for the fundamental eigenmode.

It appears to the authors that the primary motivation behind the reduction method presented in Ref. 2 was to approximate the fundamental eigenvalue of non-self-adjoint systems, and this should be apparent from the example problems in the reference, where different methods are compared with respect to the fundamental eigenvalue when the design variables are perturbed by small quantities. As Canfield correctly pointed out, the underlying choice of basis vectors is the same in both this method and our own work. However, as noted by Canfield, the criteria used for selecting the best approximation are quite different. Reference 2 advocates the selection of the root closest to the first-order Taylor series (TSA1). We expect this method to work well only for cases when the design variable perturbations are small. In contrast to Ref. 2, the root-selection criterion in Ref. 1 uses the Rayleigh quotient argument to select the best root for the fundamental mode and proposes five heuristic criteria that ensure good approximations for the higher eigenmodes. This is the point of departure between Refs. 1 and 2. Note that the second criterion in our method uses the zero-order Rayleigh quotient and not TSA1, as incorrectly mentioned by Canfield.

Further, an important contribution of Ref. 1 was to demonstrate that this method can be used to approximate both eigenvalues and eigenvectors for moderate to high rank perturbations in the system matrices. The accuracy of the approximations depends on both the rank and the magnitude of the perturbations. In contrast, because the focus of Ref. 2 was on small perturbations, such trends were not reported in that study.

Canfield's suggestion of using TSA2 in the root-selection criterion works well for the example problems taken up in Ref. 1. In fact, we observed that this procedure gives the same root as the strategy based on the five heuristic criteria. However, the authors' current view says that it would be more appropriate to choose the root closer to the higher-order perturbation of the eigenvalue derived in Ref. 4 as

$$\hat{\lambda}_i = \lambda_i^o + \frac{\phi_i^{oT}\big(\Delta \boldsymbol{K} - \lambda_i^o \Delta \boldsymbol{M}\big)\big(\phi_i^o + \Delta\phi_i\big)}{\phi_i^{oT}(\boldsymbol{M}^o + \Delta \boldsymbol{M})\big(\phi_i^o + \Delta\phi_i\big)} \tag{1}$$

where the notations used are the same as followed in Ref. 1. This suggestion is made here on the basis that this expression generally gives better quality approximations as compared to TSA2.

Canfield is not strictly correct in pointing out that the method in Ref. 1 implicitly gives the TSA2 for the eigenvector and a third-order approximation for the eigenvalue. The order of the approximation depends on both the definition of approximation order and how the basis vectors are computed. The term $\Delta\phi$ could be computed using either the first-order eigenvector derivatives or a matrix power series.[5] The latter procedure would be more efficient for cases involving a large number of design variables. Results obtained from both of these choices of the basis vectors are equivalent only when the system matrices are a linear function of the design variables under consideration. Note that the approximation order derived in Ref. 2 was based on the assumption that the perturbed system matrices and the basis vectors are computed via a Taylor series. It is also important to note that, as indicated by the results of Ref. 1, the approximation order of a method is not a good indicator of the accuracy for cases involving large perturbations.

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